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COMMENT

Directed travelling salesman problem

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Abstract. We consider here an exactly soluble directed travelling salesman problem, where the salesman is forbidden to move, during its visit to the cities, opposite to a particular direction. When the cities are randomly distributed, with concentration p, on the sites of a square lattice of linear size L, the optimised total contour length becomes $(1/p)+[2(2p-1)/p^2]\,(1/L)$ per city ($p>0$).

Optimisation problems arise naturally in various contexts ranging from designs of integrated or printed circuits (where one optimises the total connecting length between the various circuit components) to statistical physics (see, e.g., Kirkpatrick 1984, Siarry and Dreyfus 1984, Kirkpatrick and Toulouse 1985). In fact, simulated annealing with a Metropolis type algorithm for many-body systems at finite temperature (annealed down to zero temperature) has been very effectively utilised to find numerically and quantitatively the near-global optimised configurations in computer aided designs (Kirkpatrick 1984, Siarry and Dreyfus 1984). Quantitative analysis of the simulated annealing algorithm requires, of course, problems simpler than the physical design of computers, and, in this context, the travelling salesman problem has received the most attention (Kirkpatrick *et al* 1983, Kirkpatrick 1984, Kirkpatrick and Toulouse 1985). However, except for the exact enumeration solution for the optimised path in small systems (Crowder and Padberg 1980) and the exact solution of a thermodynamic model which involves travelling salesman type optimisation (Vannimenus and Mézard 1985), all the available solutions to the problem are, so far, numerically obtained using Monte Carlo type simulations or simulated annealing (Kirkpatrick 1984, Kirkpatrick and Toulouse 1985).

We consider here a directed version of the travelling salesman problem, which is very simple and is exactly soluble. In fact, all the lattice statistical problems become simple, or at least easier to study, when a directedness is introduced; see e.g., Kinzel and Yeomans (1981) for directed percolation, Dhar (1982) for the directed animal problem and Chakrabarti and Manna (1983) and Redner and Majid (1983) for directed self-avoiding walk statistics.

Let us assume that the cities (occupied sites) are distributed randomly, with concentration p, on a square lattice of linear size L. In the directed travelling salesman problem, the salesman is forbidden to move opposite to one fixed lattice direction (say, the upward direction in figure 1) during its visit to the cities. The salesman then visits each city of a given (horizontal) layer and goes to the extreme end city (occupied site) in that layer or of the next layer, and then moves down to the next layer. None

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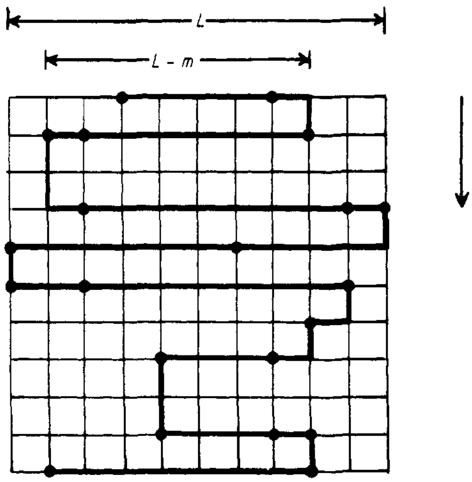

Figure 1. A typical finite section (10 × 10) of the square lattice containing 21 sites (cities), shown by full circles, distributed randomly with a concentration $p = 0.21$. The salesman never moves opposite to the fixed lattice direction shown. The thick line shows the directed travelling salesman's path. For notation, see the text.

of the lattice bonds are traced twice and the travel is restricted along the lattice bonds (streets) only (see figure 1). After visiting all the cities, down to the last layer, the salesman comes back straight to the origin. The solubility of this problem comes from the non-degeneracy of the 'Manhattan type' path traced by the directed salesman, compared to those of an undirected travelling salesman, where, similar to the spin glass problem, the degeneracy of the ground state makes the problem of finding globally optimised configurations difficult.

The total contour length for the directed travelling salesman is then given by (see figure 1)

$$S = Lp^2 \sum_{l=0}^{L} q^m (m+1)(L-m) + 2L, \tag{1}$$

where $q = (1-p)$. This is so because, if in a layer the two extreme occupied sites are separated by a distance l, then $m = L - l$ sites are vacant (each with probability q) in that layer, on either side (two sides together) of the occupied sites, and they can be arranged in $(m+1)$ ways. The extra $2L$ term comes from the downward travel after the visit of each layer and the final upward travel (direct) to the first layer. (For return to the origin, there would be a small correction term which would disappear in the large p and L limit.)

For $q < 1$ $(p > 0)$, one then immediately obtains

$$\begin{aligned} S = {} & L^2[q(1-q^{L+1}) + p(1-q^{L+1}) - (L+1)pq^{L+1}] \\ & + L[2 - 2q(1-q^{L+1}) - (2/p)q^2(1-q^{L+1}) \\ & + (L+2)(L+1)pq^{L+1} + 2(L+1)q^{L+2}], \end{aligned} \tag{2}$$

which, in the large L limit $(p > 0)$, becomes

$$S = L^2 + [2(2p-1)/p]L. \tag{3}$$

This is correct for any $p > 0$ (more specifically for $(1-p)^L \to 0$) and gives, in the $p = 1$ limit, the contour length of magnitude $S = L^2 + 2L$, which is exact for the undirected problem also.

A mean field type argument for the ordinary travelling salesman problem would indicate that the average travel distance s between two cities, on a two-dimensional lattice, would be of the order of $(L^2/pL^2)^{1/2} \sim 1/(p)^{1/2}$, while in the directed travelling salesman problem $s = S/pL^2$ is equal to $(1/p)+[2(2p-1)/p^2](1/L)$. Thus the directed optimisation contributes an extra factor $(O(1/p^{1/2}))$ in the leading-order term in the average optimised length (which becomes prominent as $p \to 0$; the limit $L \to \infty$, $p \to 0$, L^2p = constant, corresponds to the continuum limit) with a $(1/L)$ order correction term. It may be noted that, at $p = \frac{1}{2}$, the $(1/L)$ order correction term vanishes because the negative gain of order L over the L^2 order term in the total optimised travel length in (3) cancels exactly the vertical downward and upward travel length.

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Note added in proof. The path length $S - 2L$, discussed here, is the sum of the distances between two extreme end occupied cities in each layer. The salesman has to move an extra length, coming from the mismatch of the extreme end cities in two successive layers. Taking this correction $2Lq/(1-q^2)$ into account, the total path length S in (3) becomes $S = L^2 + 2[1 - q^2/(1-q^2)]L$, giving $s = 1/p + [2(1-2q^2)/(1-q^2)p](1/L)$. This correction and some other exact solutions of the undirected problem, e.g. on the Bethe lattice, will be discussed elsewhere (Barma *et al* 1986 to be published).

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